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Gravitational Gauge Theory Developed Based on the Stephenson-Kilmister-Yang Equation

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Abstract A Yang-Mills formulation of Einstein gravity with spin-affine connection as the dynamical variable of gravitational field is suggested based on the Stephenson-Kilmister-Yang (SKY) equation. A physically interesting property of the present formalism is that the Einstein field equation appears as a first-integral solution to the Yang-Mills type gravitational gauge field equation. The gravitational current density, the law of conservation and the gravitational gauge field strength in vierbein formulation are discussed. The present scheme could provide us with new insight into a possible way to include both Yang-Mills field and gravitational gauge field into one framework of generalized vierbein fields.

Keywords Vierbeins · Spin connection · Stephenson-Kilmister-Yang equation · First-integral solution

1 Introduction

With the development of quantum field theories and the increasing evidences from observational cosmology [1–3], much work has been performed in order to suggest extended theories of gravitation and to interpret new gravitational phenomena and effects [4–6]. These included, e.g., the running Newtonian gravitational constant on the cosmological-horizon scale as a modification to Einstein gravity [5], the chameleon-field model (compatible with present fifth-force experiments and cosmological observations) [6], the gravity of antisymmetric skewon field (which, together with symmetric metric, forms Hermitian fiber metric) [7], various versions of gravity with torsion [8, 9], $1/R$-correction gravity [10], and a

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variety of gauge approaches to gravitation [11]. In this paper, we suggest a gravitational gauge theory using the spin-affine connection as a gauge field connection. By means of the gauge approach, the Einstein gravity theory as a field theory of Yang-Mills type will be reformulated. In the literature, though many references have pointed out that the gravitational field is a non-Abelian gauge field [12–15], yet the field equation in general relativity (GR) is not a Yang-Mills type equation. We find that the reason why the local Lorentz symmetry in the formulation of both metric and Levi-Civita connection does not allow to describe gravity as a Yang-Mills type gauge interaction is because these two gauge interactions are formulated in different languages: specifically, GR is constructed using Levi-Civita connection (in terms of the metric), while the Yang-Mills gauge field theory is described using non-Abelian gauge field connection, which can actually be expressed in terms of the so-called 'Yang-Mills vielbein' (this would be addressed in this paper). In other words, in the Yang-Mills gauge theory, the connection and the fundamental dynamical variable are the same physical quantities, while in GR the connection (Levi-Civita connection) and the fundamental dynamical variable (metric) are not the same quantities. It is possible, in fact, to demonstrate that this asymmetry between GR and Yang-Mills gauge field theory can be avoided if we introduce the formulation of vielbein (in four dimensions we would call it a vierbein) for gravity. Here, we will verify that in the formulation of vierbein, where the spin-affine connection is written in terms of the vierbein fields, one can identify the spin-affine connection with the dynamical variable of a non-Abelian gauge field, and GR can then be reformulated as a gauge field theory of Yang-Mills type, where the field equation is suggested based on the local Lorentz-group gauge invariance with the spin-affine connection involved in the Yang-Mills covariant derivatives (in the vierbein formulation).

The present gravitational gauge formalism is developed based on the Stephenson-Kilmister-Yang theory, which contains a vacuum gravitational field equation of Yang-Mills type with Levi-Civita connection being the dynamical variable. One of the most important properties of this equation is such that it is a third-order differential equation of metric but second-order equation of Levi-Civita connection. In 1974 Yang considered an integral formalism of gauge fields and suggested such a new gravitational field equation [16], where the Christoffel symbol (Levi-Civita connection) serves as a non-Abelian gauge field [16]. Pavelle immediately pointed out that this gravitational field equation is identical with that proposed by Kilmister in 1959 [17], and then in the references published later, some authors referred to it as the Kilmister-Yang equation [18]. But actually, it might be Stephenson who was the first one to propose such a kind of theory (even one year earlier than Kilmister did) [19]. Thus the Kilmister-Yang equation can also be referred to as the Stephenson-Kilmister-Yang (SKY) equation. In general, the source-free SKY field equation can be written as

$$\nabla_\mu R^\mu{}_{\nu\alpha\beta} = 0, \tag{1}$$

or equivalently in the form $\nabla_\alpha R_{\beta\nu} - \nabla_\beta R_{\alpha\nu} = 0$. It can be readily verified that the Einstein vacuum field equation has already been involved in (1). However, the SKY equation has some other new solutions, which were viewed as unphysical solutions [20]. It was suggested that the SKY equation should be supplemented by further restrictions on the class of allowable spacetime in order to rule out the so-called unphysical solutions [20]. Later, the SKY vacuum field equation received increasingly more attentions from researchers. For example, the SKY equation for the geometrically degenerate cases of conformally flatness and decomposability of spacetime was studied [18, 21], and the possible unphysical metrics [20] that belong to these degenerate classes [18, 21] were considered. Some specific physical properties such as the monopole gravitational radiation and the Birkhoff theorem relevant to the SKY equation were discovered [22]. The static, spherically symmetric solution to the SKY equation was obtained, which showed that the Solar experiments cannot yet distinguish between the SKY equation and the Einstein vacuum gravitational equation [17].

Though it has some "unphysical" solutions, the SKY equation has already contained all the solutions of the Einstein vacuum field equation. Therefore, the SKY equation deserves further investigations, since it can be said to be more general than the Einstein field equation. We shall generalize the SKY gravitational field equation of vacuum to the case with sources (matter fields), and then rewrite it as a Yang-Mills type gravitational gauge field equation in the vierbein formulation, where the spin-affine connection (or spin connection, for brevity) serves as the Yang-Mills dynamical variable. The present prescription complies with the local Lorentz-group gauge symmetry. In the conventional gravity theory, it is the metric that serves as the dynamical variable. In the present formalism, however, we have a new dynamical equation (i.e., a Yang-Mills type gravitational gauge field equation) with a new dynamical variable (i.e., spin-affine connection). It should be noted that in the literature Camenzind has suggested such a kind of theory, where the Lorentz frame bundle on the spacetime acts as gauge bundle and the Lorentz connection (spin connection) as gauge connection, and presented a Yang-Mills type gravitational field equation with source of matter [23–25]. In this paper, however, the cosmological constant term (besides the matter term) is also taken into account. By using the Yang-Mills type gauge field equation of gravity, we can suggest a different viewpoint about the physical meanings of the cosmological constant, and provide new insight to look at the cosmological constant problem (since the Yang-Mills gravitational field equation is a third-order differential equation of metric, the covariant derivative of the cosmological constant term expressed by the vierbeins such as $-(\mathrm{i}\lambda/4)(\vartheta^{\mu}{}_{q}\vartheta^{\nu}{}_{p}-\vartheta^{\mu}{}_{p}\vartheta^{\nu}{}_{q})$ unavoidably vanishes). In addition, a new concept (Yang-Mills vielbein) will also be defined and the vielbein version for the Yang-Mills gauge field theory will be suggested, and then the spin-connection gravitational gauge theory can be generalized to the case that includes both gravitation and Yang-Mills gauge interactions.

2 The SKY Equation with Source and the Spin-Connection Gauge Field in the Vierbein Formulation

In order to suggest a Yang-Mills type field equation of gravitation, we should first reformulate the Riemannian geometry and torsionless gravity using the formulation of vierbein, where the vierbein fields satisfy the relations $g_{\mu\nu}=\vartheta_{\mu}{}^{r}e_{r\nu}$ and $\delta_{\mu}{}^{\nu}=\vartheta_{\mu}{}^{r}e_{r}{}^{\nu}$. If the metric $g_{\mu\nu}$ (viewed as the element of a matrix $\mathbf{g}$) is a complex Hermitian metric (i.e., $g_{\mu\nu}$ has a symmetric real part and an antisymmetric imaginary part), then one can show that $e_{r\nu}=(\vartheta_{\nu r})^{*}$ and $\vartheta_{\mu}{}^{r}=(e^{r}{}_{\mu})^{*}$, where the asterisk denotes the complex conjugate. Here, the Greek and Latin indices denote the Einstein local coordinate indices and the spacetime indices of local inertial frame (Lorentz coordinate indices), respectively. A gravity theory can be established based on the complex Hermitian-metric Riemannian geometry. In this theory, the nonzero torsion tensor must appear, since the contorsion is no longer a tensor and cannot be taken to be zero. In the gravity theory with real symmetric metric, however, the contorsion is a tensor, and then both the contorsion and the torsion can be assumed to be zero. In general, we can suggest a Yang-Mills type gravity theory in the vierbein formulation, where the complex Hermitian-metric Riemannian spacetime exhibits its nonzero torsion. But as a tentative study, here we concentrate our attention only on the curvature-only theory, where $g_{\mu\nu}$ is a real Hermitian metric (with vanishing imaginary part) and the contorsion is taken to be zero (besides, the metric is assumed to be analytic. Gravitomagnetic monopole would lead to the non-analytic metric [26, 27], which we will not consider, for simplicity). Under this condition, one can arrive at $e_{r\nu}=\vartheta_{\nu r}$, $\vartheta_{\nu}{}^{r}=e^{r}{}_{\nu}$ (and hence we have $\vartheta^{\nu}{}_{r}=e_{r}{}^{\nu}$ and $e^{r\nu}=\vartheta^{\nu r}$) as well as the relations between the metric and the vierbeins: $g_{\mu\nu}=e^{r}{}_{\mu}e_{r\nu}$ and $\delta_{\mu}{}^{\nu}=e^{r}{}_{\mu}e_{r}{}^{\nu}$. In this paper, we choose the metric sign convention $(+---)$. In the Riemannian geometry, the Levi-Civita covariant derivatives of the vierbeins are defined through $\nabla_{\lambda}\vartheta_{\nu r}=\partial_{\lambda}\vartheta_{\nu r}-\vartheta_{\sigma r}\Gamma^{\sigma}{}_{\lambda\nu}$ and $\nabla_{\lambda}\vartheta^{\nu}{}_{r}=\partial_{\lambda}\vartheta^{\nu}{}_{r}+\Gamma^{\nu}{}_{\lambda\sigma}\vartheta^{\sigma}{}_{r}$. Then it follows that the Levi-Civita affine connection can be expressed in terms of the vierbein fields:

$$\Gamma^{\mu}{}_{\lambda\nu}=\vartheta^{\mu r}\partial_{\lambda}\vartheta_{\nu r}+S^{\mu}{}_{\lambda\nu}, \tag{2}$$

where $S^{\mu}{}_{\lambda\nu}=\vartheta_{\nu r}\nabla_{\lambda}\vartheta^{\mu r}$ (∇_{λ} denotes the Levi-Civita covariant derivative).

Now we rewrite the Riemannian curvature tensor (using the vierbein formulation) in terms of the spin-connection gauge field, and then rewrite the Einstein gravitational field equation as a version of Yang-Mills type. We start the derivation from the Bianchi identity. We contract the coordinate index μ with γ in the Bianchi identity $\nabla_{\gamma}R_{\alpha\beta}{}^{\mu}{}_{\nu}+\nabla_{\beta}R_{\gamma\alpha}{}^{\mu}{}_{\nu}+\nabla_{\alpha}R_{\beta\gamma}{}^{\mu}{}_{\nu}=0$, and obtain $\nabla_{\mu}R_{\alpha\beta}{}^{\mu}{}_{\nu}+\nabla_{\beta}R_{\mu\alpha}{}^{\mu}{}_{\nu}+\nabla_{\alpha}R_{\beta\mu}{}^{\mu}{}_{\nu}=0$. By using the definition of Ricci tensor $R_{\alpha\nu}=R_{\mu\alpha}{}^{\mu}{}_{\nu}$, $R_{\beta\nu}=-R_{\beta\mu}{}^{\mu}{}_{\nu}$, the Bianchi identity becomes $\nabla_{\mu}R_{\alpha\beta}{}^{\mu}{}_{\nu}=\nabla_{\alpha}R_{\beta\nu}-\nabla_{\beta}R_{\alpha\nu}$. By introducing the vierbein fields, one can rewrite the Bianchi identity as the form

$$\nabla_{\mu}R^{rs\mu}{}_{\nu}=\left(\nabla_{\mu}R_{\alpha\beta}{}^{\mu}{}_{\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}+R_{\alpha\beta}{}^{\mu}{}_{\nu}\nabla_{\mu}\left(\vartheta^{\alpha r}\vartheta^{\beta s}\right). \tag{3}$$

We shall now obtain the relations that relate $\nabla_{\mu}\vartheta^{\alpha r}$ and $\nabla_{\mu}\vartheta^{\beta s}$ to the spin connections: (i) by using the relation $S_{\lambda\mu}{}^{\alpha}=(\nabla_{\mu}\vartheta_{\lambda s})\vartheta^{\alpha s}=-\vartheta_{\lambda s}\nabla_{\mu}\vartheta^{\alpha s}$, we can obtain $\vartheta^{\lambda r}S_{\lambda\mu}{}^{\alpha}=-\vartheta^{\lambda r}\vartheta_{\lambda s}\nabla_{\mu}\vartheta^{\alpha s}\Rightarrow S^{r}{}_{\mu}{}^{\alpha}=-\nabla_{\mu}\vartheta^{\alpha r}$; (ii) by using the definition $S^{\beta}{}_{\mu}{}^{\nu}=(\nabla_{\mu}\vartheta^{\beta r})\vartheta^{\nu}{}_{r}$, we can arrive at $S^{\beta}{}_{\mu}{}^{\nu}\vartheta_{\nu}{}^{s}=(\nabla_{\mu}\vartheta^{\beta r})\vartheta^{\nu}{}_{r}\vartheta_{\nu}{}^{s}\Rightarrow S^{\beta}{}_{\mu}{}^{s}=\nabla_{\mu}\vartheta^{\beta s}$. Then, based on the following two important relations

$$\nabla_{\mu}\vartheta^{\alpha r}=-S^{r}{}_{\mu}{}^{\alpha},\qquad \nabla_{\mu}\vartheta^{\beta s}=S^{\beta}{}_{\mu}{}^{s}, \tag{4}$$

the second term on the right-handed side of the identity (3) becomes

$$\begin{aligned}
R_{\alpha\beta}{}^{\mu}{}_{\nu}\nabla_{\mu}\left(\vartheta^{\alpha r}\vartheta^{\beta s}\right)&=R_{\alpha\beta}{}^{\mu}{}_{\nu}\left(\nabla_{\mu}\vartheta^{\alpha r}\right)\vartheta^{\beta s}+R_{\alpha\beta}{}^{\mu}{}_{\nu}\vartheta^{\alpha r}\nabla_{\mu}\vartheta^{\beta s}\\
&=-R_{\alpha\beta}{}^{\mu}{}_{\nu}S^{r}{}_{\mu}{}^{\alpha}\vartheta^{\beta s}+R_{\alpha\beta}{}^{\mu}{}_{\nu}\vartheta^{\alpha r}S^{\beta}{}_{\mu}{}^{s}\\
&=-S^{r}{}_{\mu t}R^{ts\mu}{}_{\nu}+R^{rt\mu}{}_{\nu}S_{t\mu}{}^{s}\\
&=-[S_{\mu},\Omega^{\mu}{}_{\nu}]^{rs},
\end{aligned} \tag{5}$$

where we have substituted the relations $S^{r}{}_{\mu}{}^{\alpha}=S^{r}{}_{\mu t}\vartheta^{\alpha t}$ and $S^{\beta}{}_{\mu}{}^{s}=\vartheta^{\beta t}S_{t\mu}{}^{s}$. The definitions $R^{ts\mu}{}_{\nu}=R_{\alpha\beta}{}^{\mu}{}_{\nu}\vartheta^{\alpha t}\vartheta^{\beta s}$, $R^{rt\mu}{}_{\nu}=R^{r}{}_{\beta}{}^{\mu}{}_{\nu}\vartheta^{\beta t}$ and $R^{ts\mu}{}_{\nu}\equiv(\Omega^{\mu}{}_{\nu})^{ts}$, $R^{rt\mu}{}_{\nu}\equiv(\Omega^{\mu}{}_{\nu})^{rt}$ have been used in (5). In the following analysis, the definitions of the spin connection as well as the Riemannian curvature tensor of the vierbein formulation, such as $(S_{\mu})^{r}{}_{t}\equiv S^{r}{}_{\mu t}$, $(S_{\mu})_{t}{}^{s}\equiv S_{t\mu}{}^{s}$, and $(\Omega^{\mu}{}_{\nu})^{rs}\equiv R^{rs\mu}{}_{\nu}=\vartheta^{\alpha r}R_{\alpha\beta}{}^{\mu}{}_{\nu}\vartheta^{\beta s}$, will always be adopted for derivations. Based on the above relations, it follows from (3) and (5) that the Bianchi identity becomes

$$\nabla_{\mu}(\Omega^{\mu}{}_{\nu})^{rs}=\left(\nabla_{\alpha}R_{\beta\nu}-\nabla_{\beta}R_{\alpha\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}-[S_{\mu},\Omega^{\mu}{}_{\nu}]^{rs}. \tag{6}$$

By using the Einstein gravitational field equation $R_{\beta\nu}=\kappa\hat{T}_{\beta\nu}$, one can arrive at

$$\nabla_{\mu}(\Omega^{\mu}{}_{\nu})^{rs}+[S_{\mu},\Omega^{\mu}{}_{\nu}]^{rs}=\kappa\left(\nabla_{\alpha}\hat{T}_{\beta\nu}-\nabla_{\beta}\hat{T}_{\alpha\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}. \tag{7}$$

Here, the source tensor $\hat{T}_{\beta\nu}$ has a form $\hat{T}_{\beta\nu} = \tau_{\beta\nu} - g_{\beta\nu}\tau/2 - (\lambda/\kappa)g_{\beta\nu}$ (in four-dimensional spacetime) or has a general form

$$\hat{T}_{\beta\nu} = \tau_{\beta\nu} - \frac{1}{2}\left(\frac{1}{\frac{d}{2}-1}\right)g_{\beta\nu}\tau - \frac{\lambda}{\kappa}\left(\frac{1}{\frac{d}{2}-1}\right)g_{\beta\nu} \tag{8}$$

(in d-dimensional spacetime), where $\tau_{\beta\nu}$ denotes the energy-momentum tensor of matter fields (the scalar $\tau = g^{\beta\nu}\tau_{\beta\nu}$) and λ stands for the cosmological constant. We are now in a position to study the property of the spin connection $(S_\mu)^{rs}$ (i.e. $S^r{}_\mu{}^s$). As $S^r{}_\mu{}^s$ is antisymmetric in indices r and s, i.e., $S^r{}_\mu{}^s = -S^s{}_\mu{}^r$, we have $(S^\dagger_\mu)^{rs} = -(S_\mu)^{rs}$, $S^\dagger_\mu = -S_\mu$, where the Lorentz indices r, s are regarded as the matrix indices. If we define a new quantity $\check{S}_\mu = \mathrm{i}S_\mu$, then we have $(\check{S}^\dagger_\mu)^{rs} = (\check{S}_\mu)^{rs}$, $\check{S}^\dagger_\mu = \check{S}_\mu$. This means that $\check{S}_\mu$ is a Hermitian spin-connection non-Abelian gauge field potential (dynamical variable). Hence, (7) can be rewritten as a Yang-Mills type gauge field equation

$$\nabla_\mu(\Omega^\mu{}_\nu)^{rs} - \mathrm{i}[\check{S}_\mu, \Omega^\mu{}_\nu]^{rs} = \kappa\left(\nabla_\alpha\hat{T}_{\beta\nu} - \nabla_\beta\hat{T}_{\alpha\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}. \tag{9}$$

Now let us see whether $(\Omega^\mu{}_\nu)^{rs}$ could be expressed in terms of the spin-connection non-Abelian gauge field potential (i.e., whether it can be reformulated as a standard Yang-Mills gauge field strength). This is not a new problem, since the authors of many references have considered it from various aspects [12–15]. By using the definition of Levi-Civita covariant derivative, the second-order covariant derivative of the vierbein $\vartheta_\tau{}^r$ yields a relation

$$\begin{aligned}&\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu} = \vartheta_\beta{}^r R^\beta{}_{\tau\mu\nu}\\ \Rightarrow\quad &\left(\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu}\right)\vartheta^{\tau s} = \vartheta_\beta{}^r R^\beta{}_{\tau\mu\nu}\vartheta^{\tau s},\end{aligned} \tag{10}$$

which can be rewritten as

$$(\Omega_{\mu\nu})^{rs} = \left(\vartheta_\tau{}^r{}_{;\mu;\nu} - \vartheta_\tau{}^r{}_{;\nu;\mu}\right)\vartheta^{\tau s}. \tag{11}$$

With the help of the relations in (4), we can obtain

$$\vartheta_\tau{}^r{}_{;\mu;\nu}\vartheta^{\tau s} = \nabla_\nu S^s{}_\mu{}^r + (S_\mu S_\nu)^{rs}, \qquad \vartheta_\tau{}^r{}_{;\nu;\mu}\vartheta^{\tau s} = \nabla_\mu S^s{}_\nu{}^r + (S_\nu S_\mu)^{rs}. \tag{12}$$

Thus the Riemannian curvature tensor in the formulation of vierbein is of the form $(\Omega_{\mu\nu})^{rs} = [\nabla_\nu S^s{}_\mu{}^r + (S_\mu S_\nu)^{rs}] - [\nabla_\mu S^s{}_\nu{}^r + (S_\nu S_\mu)^{rs}]$, which can be rewritten as $\nabla_\mu S^r{}_\nu{}^s - \nabla_\nu S^r{}_\mu{}^s + (S_\mu S_\nu)^{rs} - (S_\nu S_\mu)^{rs} = (\nabla_\mu S_\nu - \nabla_\nu S_\mu + [S_\mu, S_\nu])^{rs}$. Therefore, we have

$$(\Omega_{\mu\nu})^{rs} = \frac{1}{\mathrm{i}}\left(\nabla_\mu\check{S}_\nu - \nabla_\nu\check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]\right)^{rs}, \tag{13}$$

or $\Omega_{\mu\nu} = (\nabla_\mu\check{S}_\nu - \nabla_\nu\check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu])/\mathrm{i}$. It follows that a Hermitian spin-connection gauge field strength (curvature) can be defined as $\check{\Omega}_{\mu\nu} = \mathrm{i}\Omega_{\mu\nu}$. Thus, one can have $\check{\Omega}_{\mu\nu} = \nabla_\mu\check{S}_\nu - \nabla_\nu\check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]$ whose component (matrix element) is given by $(\check{\Omega}_{\mu\nu})^{rs} = (\nabla_\mu\check{S}_\nu - \nabla_\nu\check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu])^{rs}$.

Now the field equation (9) that contains the Einstein equation can be rewritten as

$$\nabla_\mu(\check{\Omega}^\mu{}_\nu)^{rs} - \mathrm{i}[\check{S}_\mu, \check{\Omega}^\mu{}_\nu]^{rs} = \kappa\mathrm{i}\left(\nabla_\alpha\hat{T}_{\beta\nu} - \nabla_\beta\hat{T}_{\alpha\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}. \tag{14}$$

Here, the current density of the source is

$$(J_\nu)^{rs} = \mathrm{i}\left(\nabla_\alpha \hat{T}_{\beta\nu} - \nabla_\beta \hat{T}_{\alpha\nu}\right)\vartheta^{\alpha r}\vartheta^{\beta s}. \tag{15}$$

Equation (14) can be rewritten as

$$\nabla_\mu \check{\Omega}^\mu{}_\nu - \mathrm{i}[\check{S}_\mu, \check{\Omega}^\mu{}_\nu] = \kappa J_\nu, \tag{16}$$

or

$$\mathcal{D}_\mu \check{\Omega}^\mu{}_\nu = \kappa J_\nu, \tag{17}$$

where the spin-connection covariant derivative $\mathcal{D}_\mu = \nabla_\mu - \mathrm{i}[\check{S}_\mu, \cdot\,]$. This is the Yang-Mills type field equation of gravitation obtained in the vierbein formulation.

In the regular prescription of gravity in GR, where the formulation of both the metric and the Levi-Civita connection is used, there are almost no similarities between GR and Yang-Mills theory seen from the mathematical structures of the field equations. We have shown (in the representation of tensor for the local Lorentz group) that the Einstein field equation of gravitation in the vierbein formulation can be reformulated as a Yang-Mills version, where the dynamical variable of the gauge field is the spin connection, and the Lorentz group is the local gauge group.

3 Gravitational Current Density and Law of Conservation

We have defined a gravitational current density by using the gravitational gauge field theory. Here we discuss the current density, $(J_\nu)^{rs}$, on the right-handed side of the new gravitational field equation (16). As the Lorentz indices r, s can be regarded as the matrix indices in the vierbein formulation, we can now consider its complex conjugate:

$$((J_\nu)^{rs})^* = \mathrm{i}\left(\nabla_\alpha \hat{T}_{\beta\nu} - \nabla_\beta \hat{T}_{\alpha\nu}\right)\vartheta^{\alpha s}\vartheta^{\beta r}, \tag{18}$$

which equals $(J_\nu)^{sr}$. Besides, it can be readily verified that, for a matrix, we have an identity $((J_\nu)^{rs})^* \equiv (J_\nu^\dagger)^{sr}$. Thus, the result of (18) means that $J_\nu^\dagger = J_\nu$, namely, J_ν can be viewed as a Hermitian current density. Therefore, we obtain a non-Abelian current density that is involved in the gravitational interaction. It can be shown that the gravitational current density, $(J_\nu)^{sr}$, of matter field can be rewritten as $(J_\nu)^{sr} = \mathrm{i}[\mathcal{D}_\alpha(\hat{T}_{\beta\nu}\vartheta^{\alpha s}\vartheta^{\beta r}) - \mathcal{D}_\beta(\hat{T}_{\alpha\nu}\vartheta^{\alpha s}\vartheta^{\beta r})]$, i.e.,

$$(J_\nu)^{sr} = \mathrm{i}\mathcal{D}_\mu(\vartheta^{\mu s}\hat{T}^r{}_\nu - \vartheta^{\mu r}\hat{T}^s{}_\nu) \equiv 2\mathcal{D}_\mu(\check{\mathcal{C}}^\mu{}_\nu)^{sr}, \tag{19}$$

where the Hermitian source tensor $(\check{\mathcal{C}}^\mu{}_\nu)^{sr}$ is defined as

$$(\check{\mathcal{C}}^\mu{}_\nu)^{sr} = \frac{\mathrm{i}}{2}\left(\vartheta^{\mu s}\hat{T}^r{}_\nu - \vartheta^{\mu r}\hat{T}^s{}_\nu\right). \tag{20}$$

It follows from (14) to (17) that the gravitational field equation can now be rewritten as a form of spin-connection covariant divergence:

$$\mathcal{D}_\mu\left[(\check{\Omega}^{\mu\nu})_{rs} - 2\kappa(\check{\mathcal{C}}^{\mu\nu})_{rs}\right] = 0. \tag{21}$$

In what follows, we deal with the problem of conservation law of the current density. In the torsion-free curvature-only differential geometry, it can be readily verified that $\nabla_\nu\nabla_\mu\check{\Omega}^{\mu\nu}$ vanishes, i.e.,

$$\begin{aligned}\nabla_\nu\nabla_\mu\check{\Omega}^{\mu\nu} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}\nabla_\mu\check{\Omega}^{\mu\nu}\right)\\ &= \frac{1}{\sqrt{-g}}\frac{\partial^2}{\partial x^\nu\partial x^\mu}\left(\sqrt{-g}\check{\Omega}^{\mu\nu}\right)\\ &\equiv 0.\end{aligned}\tag{22}$$

According to the Yang-Mills type gravitational field equations $\mathcal{D}_\mu\check{\Omega}^{\mu\nu}=\kappa J^\nu$, i.e., $\nabla_\mu\check{\Omega}^{\mu\nu}-\mathrm{i}[\check{S}_\mu,\check{\Omega}^{\mu\nu}]=\kappa J^\nu$, we can define an effective current density

$$\mathcal{J}^\nu_{\rm eff}=J^\nu+\frac{\mathrm{i}}{\kappa}[\check{S}_\mu,\check{\Omega}^{\mu\nu}].\tag{23}$$

Thus the equation of conserved current is given by

$$\nabla_\nu\mathcal{J}^\nu_{\rm eff}=0.\tag{24}$$

Equations (23) and (24) are the conserved current density and its continuity equation, respectively. Equation (24) is, however, not a Yang-Mills gauged covariant equation. As we will show in the following section, the present Yang-Mills formulation of gravity is a local Lorentz-group gauge theory, where the Yang-Mills covariant derivative contains the spin connection. For this reason, we need to construct a Yang-Mills gauged covariant equation for the conservation law of current density. The expected conservation equation would be of the form

$$\mathcal{D}_\nu J^\nu=0.\tag{25}$$

Now let us show that (25) indeed holds true in the Yang-Mills formulation of gravity. According to the Yang-Mills type gravitational field equation $\mathcal{D}_\mu\check{\Omega}^{\mu\nu}=\kappa J^\nu$, in order to prove (25), we should first establish that $\mathcal{D}_\nu\mathcal{D}_\mu\check{\Omega}^{\mu\nu}=0$ is correct. The calculation is presented as follows:

$$\begin{aligned}\mathcal{D}_\nu\mathcal{D}_\mu\check{\Omega}^{\mu\nu} &= \nabla_\nu\left(\nabla_\mu\check{\Omega}^{\mu\nu}-\mathrm{i}[\check{S}_\mu,\check{\Omega}^{\mu\nu}]\right)-\mathrm{i}\left[\check{S}_\nu,\nabla_\mu\check{\Omega}^{\mu\nu}-\mathrm{i}[\check{S}_\mu,\check{\Omega}^{\mu\nu}]\right]\\ &= \nabla_\nu\nabla_\mu\check{\Omega}^{\mu\nu}-\mathrm{i}[\nabla_\nu\check{S}_\mu,\check{\Omega}^{\mu\nu}]-\mathrm{i}[\check{S}_\mu,\nabla_\nu\check{\Omega}^{\mu\nu}]-\mathrm{i}\left[\check{S}_\nu,\nabla_\mu\check{\Omega}^{\mu\nu}\right]-\left[\check{S}_\nu,[\check{S}_\mu,\check{\Omega}^{\mu\nu}]\right]\\ &= -\mathrm{i}[\nabla_\nu\check{S}_\mu,\check{\Omega}^{\mu\nu}]-\left[\check{S}_\nu,[\check{S}_\mu,\check{\Omega}^{\mu\nu}]\right],\end{aligned}\tag{26}$$

where the relations $\nabla_\nu\nabla_\mu\check{\Omega}^{\mu\nu}=0$ (i.e., (22)) and $-\mathrm{i}[\check{S}_\mu,\nabla_\nu\check{\Omega}^{\mu\nu}]-\mathrm{i}[\check{S}_\nu,\nabla_\mu\check{\Omega}^{\mu\nu}]=0$ have been substituted. Then, according to the Jacobi identity $[\check{S}_\nu,[\check{S}_\mu,\check{\Omega}^{\mu\nu}]]+[\check{\Omega}^{\mu\nu},[\check{S}_\nu,\check{S}_\mu]]+[\check{S}_\mu,[\Omega^{\mu\nu},\check{S}_\nu]]=0$, we can have $[\check{S}_\nu,[\check{S}_\mu,\check{\Omega}^{\mu\nu}]]+[\check{S}_\nu,[\Omega^{\nu\mu},\check{S}_\mu]]=[[\check{S}_\nu,\check{S}_\mu],\check{\Omega}^{\mu\nu}]$, and then obtain a relation $[\check{S}_\nu,[\check{S}_\mu,\check{\Omega}^{\mu\nu}]]=[[\check{S}_\nu,\check{S}_\mu],\check{\Omega}^{\mu\nu}]/2$. Thus, the formula (26) becomes

$$\mathcal{D}_\nu\mathcal{D}_\mu\check{\Omega}^{\mu\nu}=-\mathrm{i}\left[\nabla_\nu\check{S}_\mu-\frac{\mathrm{i}}{2}[\check{S}_\nu,\check{S}_\mu],\check{\Omega}^{\mu\nu}\right].\tag{27}$$

Hence, one can arrive at $\mathcal{D}_\nu \mathcal{D}_\mu \check{\Omega}^{\mu\nu} = -(\mathrm{i}/2)[\check{\Omega}_{\nu\mu}, \check{\Omega}^{\mu\nu}] = 0$. This establishes that the covariant continuity equation (25) is really valid in the formulation of gravity as a Yang-Mills type gauge interaction.

On the other hand, the covariant continuity equation (25) can also be derived from the conservation equation (24) of the effective current. The derivation procedure is given as follows:

$$\begin{aligned}\nabla_\nu \mathcal{J}^\nu_{\mathrm{eff}} &= \nabla_\nu J^\nu + \frac{\mathrm{i}}{2\kappa}\left[\nabla_\nu \check{S}_\mu - \nabla_\mu \check{S}_\nu - \mathrm{i}[\check{S}_\nu, \check{S}_\mu], \check{\Omega}^{\mu\nu}\right] - \frac{1}{2\kappa}\left[[\check{S}_\nu, \check{S}_\mu], \check{\Omega}^{\mu\nu}\right] \\ &\quad + \frac{\mathrm{i}}{\kappa}[\check{S}_\mu, \nabla_\nu \check{\Omega}^{\mu\nu}] \\ &= \nabla_\nu J^\nu - \frac{1}{2\kappa}\left[[\check{S}_\nu, \check{S}_\mu], \check{\Omega}^{\mu\nu}\right] + \frac{\mathrm{i}}{\kappa}\left[\check{S}_\mu, \nabla_\nu \check{\Omega}^{\mu\nu} - \mathrm{i}[\check{S}_\nu, \check{\Omega}^{\mu\nu}]\right] - \frac{1}{\kappa}\left[\check{S}_\mu, [\check{S}_\nu, \check{\Omega}^{\mu\nu}]\right] \\ &= \left(\nabla_\nu J^\nu - \mathrm{i}\left[\check{S}_\nu, J^\nu\right]\right) - \frac{1}{2\kappa}\left[[\check{S}_\nu, \check{S}_\mu], \check{\Omega}^{\mu\nu}\right] - \frac{1}{\kappa}\left[\check{S}_\mu, [\check{S}_\nu, \check{\Omega}^{\mu\nu}]\right] \\ &= \mathcal{D}_\nu J^\nu, \end{aligned} \tag{28}$$

where the identity $[[\check{S}_\nu, \check{S}_\mu], \check{\Omega}^{\mu\nu}]/2 + [\check{S}_\mu, [\check{S}_\nu, \check{\Omega}^{\mu\nu}]] \equiv 0$ has been inserted. From the result obtained in (28), one can show that the gauged covariant law of the conserved current density is $\mathcal{D}_\nu J^\nu = 0$.

It is very interesting that the role of the cosmological constant in the gravitational gauge theory is quite different from that in the Einstein gravity theory. The gravitational effect of the cosmological constant term that can be expressed as $-(\mathrm{i}\lambda/4)(\vartheta^\mu{}_q \vartheta^\nu{}_p - \vartheta^\mu{}_p \vartheta^\nu{}_q)$ is automatically eliminated in (21) because its spin-connection covariant derivative vanishes (i.e. $\mathcal{D}_\mu(\vartheta^\mu{}_q \vartheta^\nu{}_p - \vartheta^\mu{}_p \vartheta^\nu{}_q) \equiv 0$). This, therefore, means that the quantum vacuum energy (governed in the gravitational field equation by a cosmological term) would actually make no contributions to gravitation in the present Yang-Mills formulation of gravitation. This can also be interpreted in an alternative but equivalent way: there is a term $-(\mathrm{i}\lambda/2)(\check{\Omega}_{\mu\nu})^{pq} \vartheta^\mu{}_q \vartheta^\nu{}_p$ in the interaction Lagrangian density $\mathcal{L}_{\mathrm{int}}$. This term can be rewritten as $-\lambda R/2$. It should be noted that the variation of $-\lambda R/2$ with respect to the spin-affine connection vanishes because $-\lambda R/2$ is a linear function of the spin-affine connection. This will, unavoidably, change our understandings of the physical meanings and the roles of the cosmological constant in gravitational interactions.

4 Spin-Connection Covariant Derivatives of Tensor and Spinor

The local Lorentz-group Yang-Mills type gauge symmetry for the spin-connection gravity can be easily revealed in the formulation of vierbein. We define the Lorentz rotation (in the representation of tensor) as $U^r{}_s = \partial x'^r/\partial x^s$ and $(U^{-1})^s{}_t = \partial x^s/\partial x'^t$, which satisfy the orthogonality relation

$$U^r{}_s (U^{-1})^s{}_t = (UU^{-1})^r{}_t = \delta^r{}_t \tag{29}$$

because of $(\partial x'^r/\partial x^s)(\partial x^s/\partial x'^t) = \delta^r{}_t$. In general, the Lorentz rotation (in the representation of tensor) can be written explicitly in terms of the Lorentz group generators as the following exponential forms

$$U^{rs} = \exp\left[-\frac{\mathrm{i}}{2}\alpha_{pq}(\mathcal{J}^{pq})^{rs}\right], \qquad U^r{}_s = \exp\left[-\frac{\mathrm{i}}{2}\alpha_{pq}(\mathcal{J}^{pq})^r{}_s\right], \tag{30}$$

where the Lorentz group generators (in the representation of tensor) are given by $(\mathcal{J}^{pq})^{rs} = \mathrm{i}(\eta^{pr}\eta^{qs} - \eta^{ps}\eta^{qr})$ and $(\mathcal{J}^{pq})^{r}{}_{s} = \mathrm{i}(\eta^{pr}\eta^{q}{}_{s} - \eta^{p}{}_{s}\eta^{qr})$. Here, η^{pr} denotes the flat Minkowski metric, and $\eta^{q}{}_{s} \equiv \delta^{q}{}_{s}$. We point out that the Lorentz group elements, $U^{r}{}_{t}$ and U^{rt}, have the following properties:

(i) The Lorentz indices (i.e., the matrix indices $r, s, t \ldots$) can be raised and lowered by the flat Minkowski metric η^{ts}, η_{ts}, e.g., $U^{rs} = U^{r}{}_{t}\eta^{ts}$ and $U^{r}{}_{s} = U^{rt}\eta_{ts}$, which can be proved by using the form of Taylor series expansion of the local Lorentz group elements defined in (30).

(ii) The inverse elements of the Lorentz group can be obtained by using the relations

$$(U^{-1})^{sr} = U^{rs}, \qquad (U^{-1})_{s}{}^{r} = U^{r}{}_{s}. \tag{31}$$

These two relations can also be verified by using the procedure of Taylor series expansion.

(iii) The Lorentz transformation of a vector, say P^{r}, can be written as

$$P'^{r} = \frac{\partial x'^{r}}{\partial x^{s}} P^{s} = U^{r}{}_{s} P^{s}, \tag{32}$$

and then from (32), one can show that the infinitesimal transformation is $\delta P^{r} = -(\mathrm{i}/2)\alpha_{pq}(\mathcal{J}^{pq})^{r}{}_{s}P^{s}$. This can become $(1/2)\alpha_{pq}(\eta^{pr}P^{q} - P^{p}\eta^{qr})$, which is consistent with the infinitesimal Lorentz transformation $\delta P^{r} = \alpha^{r}{}_{q}P^{q}$, where $\alpha^{r}{}_{q}$ is an infinitesimal transformation coefficient satisfying $\alpha^{r}{}_{q} = \alpha^{rs}\eta_{sq}$ with α^{rs} being antisymmetric in indices r, s.

At this stage, we present the covariant derivatives (containing the spin connection) of tensors. It is apparent, for example, that the covariant derivatives of the two typical tensors $P^{\nu r}$ and $(P^{\nu})^{rt}$ are as follows:

$$\mathcal{D}_{\mu}P^{\nu r} = \nabla_{\mu}P^{\nu r} - \mathrm{i}(\check{S}_{\mu})^{r}{}_{s}P^{\nu s}, \qquad \mathcal{D}_{\mu}(P^{\nu})^{rt} = \nabla_{\mu}(P^{\nu})^{rt} - \mathrm{i}[\check{S}_{\mu}, P^{\nu}]^{rt}. \tag{33}$$

By using the definition of the spin connection $(S_{\mu})^{tr} = \vartheta^{\tau t}\nabla_{\mu}\vartheta_{\tau}{}^{r}$, one can show from (33) that the spin-connection covariant derivatives of the vierbeins vanish, e.g. $\mathcal{D}_{\mu}\vartheta^{\tau t} \equiv 0$ and $\mathcal{D}_{\mu}\vartheta_{\tau}{}^{t} \equiv 0$. According to the local Lorentz-group gauge transformations $P'^{\nu r} = U^{r}{}_{s}P^{\nu s}$ and $\mathcal{D}'_{\mu}P'^{\nu r} = U^{r}{}_{s}\mathcal{D}_{\mu}P^{\nu s}$ (with $\mathcal{D}'_{\mu} = \nabla_{\mu} - \mathrm{i}\check{S}'_{\mu}$), one can have

$$\begin{aligned}
&\nabla_{\mu}P'^{\nu r} - \mathrm{i}(\check{S}'_{\mu})^{r}{}_{s}P'^{\nu s} = U^{r}{}_{s}\nabla_{\mu}P^{\nu s} - \mathrm{i}U^{r}{}_{s}(\check{S}_{\mu})^{s}{}_{t}P^{\nu t},\\
&\quad\Rightarrow\quad \nabla_{\mu}\left(U^{r}{}_{t}P^{\nu t}\right) - \mathrm{i}(\check{S}'_{\mu})^{r}{}_{s}U^{s}{}_{t}P^{\nu t} = U^{r}{}_{s}\nabla_{\mu}P^{\nu s} - \mathrm{i}U^{r}{}_{s}(\check{S}_{\mu})^{s}{}_{t}P^{\nu t},\\
&\quad\Rightarrow\quad \left(\partial_{\mu}U^{r}{}_{t}\right)P^{\nu t} + U^{r}{}_{t}\nabla_{\mu}P^{\nu t} - \mathrm{i}(\check{S}'_{\mu})^{r}{}_{s}U^{s}{}_{t}P^{\nu t} = U^{r}{}_{s}\nabla_{\mu}P^{\nu s} - \mathrm{i}U^{r}{}_{s}(\check{S}_{\mu})^{s}{}_{t}P^{\nu t},
\end{aligned} \tag{34}$$

which leads to

$$\begin{aligned}
&\partial_{\mu}U^{r}{}_{t} - \mathrm{i}(\check{S}'_{\mu})^{r}{}_{s}U^{s}{}_{t} = -\mathrm{i}U^{r}{}_{s}(\check{S}_{\mu})^{s}{}_{t},\\
&\quad\Rightarrow\quad (\check{S}'_{\mu})^{r}{}_{s}U^{st} = U^{r}{}_{s}(\check{S}_{\mu})^{st} - \mathrm{i}\partial_{\mu}U^{rt},\\
&\quad\Rightarrow\quad (\check{S}'_{\mu})^{r}{}_{s}U^{st}(U^{-1})_{t}{}^{p} = U^{r}{}_{s}(\check{S}_{\mu})^{st}(U^{-1})_{t}{}^{p} - \mathrm{i}\left(\partial_{\mu}U^{rt}\right)(U^{-1})_{t}{}^{p},\\
&\quad\Rightarrow\quad (\check{S}'_{\mu})^{rp} = U^{r}{}_{s}(\check{S}_{\mu})^{st}(U^{-1})_{t}{}^{p} - \mathrm{i}\left(\partial_{\mu}U^{rt}\right)(U^{-1})_{t}{}^{p}.
\end{aligned} \tag{35}$$

This is exactly the local gauge transformation of the spin-affine connection. One can thus show that the Hermitian spin-affine connection $\check{S}'_{\mu}$ automatically obeys the local Lorentz-group gauge transformation rule. The transformation rule can also be written as (omitting the matrix indices for simplicity)

$$\check{S}'_{\mu} = U\check{S}_{\mu}U^{-1} - \mathrm{i}(\partial_{\mu}U)U^{-1}. \tag{36}$$

This means that the spin-affine connection $\check{S}_{\mu}$ transforms exactly the same as the dynamical variable of a Yang-Mills gauge field. In other words, this requires establishing a dynamics of spin-affine connection for the gravitational interactions.

The Lorentz transformation of spinor is discussed in what follows in order to construct a gravitational gauge field tensor (curvature) in the spinor representation. We shall consider the representation of spinor for the Lorentz group and Lorentz rotation. We first discuss the Lorentz transformation of spinor, and see the spin-affine connection in the representation of spinor (the relations of spin-affine connections between the two representations: spinor representation and tensor representation). We have considered the Lorentz transformation operator $U^{r}{}_{s}$ that operates on the vector and tensor, and studied the covariant derivative $\mathcal{D}_{\mu}$ of the vector and tensor. Let us now consider the Lorentz transformation operator that acts on the spinors and the corresponding covariant derivative $\mathcal{D}_{\mu}^{(\mathrm{sp})}$, namely, we shall apply the Lorentz group in the representation of spinor to the local gauge field theory.

In the absence of gravity, the global Lorentz transformations of spinor (say, ψ, $\bar{\psi}$) and vector (say, ∂_t) are of the form [28, 29]

$$\psi' = U_{\frac{1}{2}}\psi, \qquad \bar{\psi}' = \bar{\psi}U_{\frac{1}{2}}^{-1}, \qquad \partial'_{s} = U_{s}{}^{t}\partial_{t}. \tag{37}$$

Here, we discuss the global Lorentz transformation of a scalar $\bar{\psi}\gamma^{s}\partial_{s}\psi$:

$$\bar{\psi}'\gamma'^{s}\partial'_{s}\psi' = \left(\bar{\psi}U_{\frac{1}{2}}^{-1}\right)\gamma'^{s}\left(U_{s}{}^{t}\partial_{t}\right)\left(U_{\frac{1}{2}}\psi\right) = \bar{\psi}\gamma^{t}\partial_{t}\psi. \tag{38}$$

This requires $\gamma^{t} = U_{\frac{1}{2}}^{-1}\gamma'^{s}U_{\frac{1}{2}}U_{s}{}^{t}$. Of course, the Dirac matrix $\gamma'^{s} = \gamma^{s}$ (this is valid under the Lorentz transformation. This would, however, no longer hold in the Edwards transformation that is a coordinate transformation for anisotropic flat spacetimes [30]. Here, however, the discussion is limited only to the case of isotropic flat spacetimes for convenience). As is known, according to $\gamma^{t} = U_{\frac{1}{2}}^{-1}\gamma^{s}U_{\frac{1}{2}}U_{s}{}^{t}$ (and hence $\gamma^{t}(U^{-1})_{t}{}^{s} = U_{\frac{1}{2}}^{-1}\gamma^{s}U_{\frac{1}{2}}$), one can immediately obtain an important relation $U_{\frac{1}{2}}^{-1}\gamma^{s}U_{\frac{1}{2}} = U^{s}{}_{t}\gamma^{t}$. Besides, it is well known that the Lorentz group generator in the representation of spinor is given by $\Sigma^{mn} = (\mathrm{i}/4)[\gamma^{m}, \gamma^{n}]$, which satisfies the commutation rule of the Lorentz algebra $[\Sigma^{mn}, \Sigma^{pq}] = \mathrm{i}(\eta^{np}\Sigma^{mq} - \eta^{mp}\Sigma^{nq} - \eta^{nq}\Sigma^{mp} + \eta^{mq}\Sigma^{np})$. By using the commutation relation $[\gamma^{p}, \Sigma^{mn}] = (\mathcal{J}^{mn})^{p}{}_{t}\gamma^{t}$ [28] and the relation $U_{\frac{1}{2}}^{-1}\gamma^{s}U_{\frac{1}{2}} = U^{s}{}_{t}\gamma^{t}$, one can obtain the Lorentz transformation operator in the representation of spinor as follows

$$U_{\frac{1}{2}} = \exp\left(-\frac{\mathrm{i}}{2}\alpha_{mn}\Sigma^{mn}\right). \tag{39}$$

Note that here the parameter α_{mn} is independent of the spacetime coordinates, i.e., $U_{\frac{1}{2}}$ in (39) is a global Lorentz transformation operator. If we consider the local Lorentz transformation, then $\alpha_{mn} = \alpha_{mn}(x^{\mu})$, and the ordinary partial derivative of spinor should be generalized to the covariant derivative that contains the compensating spin-affine connection gauge field B_μ, i.e., $\mathcal{D}_\mu^{(\mathrm{sp})} = \partial_\mu - \mathrm{i}B_\mu$.

In general, as a dynamical variable of gauge field in the Yang-Mills field theory, B_μ (the spin connection in the representation of spinor) is a fundamental physical quantity. In the vierbein formulation for the Yang-Mills field theory, however, the fundamental quantities, such as the gauge potential B_μ, could be constructed explicitly in terms of the vierbein fields. One can, for example, choose the form $B_\mu = \sigma(\check{S}_\mu)^{pq}\Sigma_{qp}$, where σ is to be determined by the spin-connection gauge transformation $B'_\mu = U_{\frac{1}{2}} B_\mu U_{\frac{1}{2}}^{-1} - \mathrm{i}(\partial_\mu U_{\frac{1}{2}})U_{\frac{1}{2}}^{-1}$. It can be verified that the parameter $\sigma = \mathrm{i}/2$. In a word, after a considerable amount of computation, one can immediately arrive at

$$\mathcal{D}_\mu^{(\mathrm{sp})}\psi = \left[\partial_\mu - \mathrm{i}\frac{\mathrm{i}}{2}(\check{S}_\mu)^{pq}\Sigma_{qp}\right]\psi, \tag{40}$$

i.e. $\mathcal{D}_\mu^{(\mathrm{sp})}\psi = [\partial_\mu + (1/2)(\check{S}_\mu)^{pq}\Sigma_{qp}]\psi$. Now let us calculate the gauge field tensor (strength, curvature) $B_{\mu\nu}$ (which is defined by $B_{\mu\nu} = \mathrm{i}[\mathcal{D}_\mu^{(\mathrm{sp})}, \mathcal{D}_\nu^{(\mathrm{sp})}]$) associated with the local Lorentz-group gauge transformation in the spinor representation: $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - \mathrm{i}[B_\mu, B_\nu]$. It can be verified that the relation between the gauge field strengths $B_{\mu\nu}$ (in the spinor representation) and $(\check{\Omega}_{\mu\nu})^{rs}$ (in the tensor representation) due to the local Lorentz rotation is given by

$$B_{\mu\nu} = \frac{\mathrm{i}}{2}(\check{\Omega}_{\mu\nu})^{rs}\Sigma_{sr}. \tag{41}$$

In what follows, we present a proof for the relation (41):

$$\begin{aligned}\partial_\mu B_\nu - \partial_\nu B_\mu - \mathrm{i}[B_\mu, B_\nu] = {}& \frac{\mathrm{i}}{2}\left[\partial_\mu(\check{S}_\nu)^{rs} - \partial_\nu(\check{S}_\mu)^{rs}\right]\Sigma_{sr} \\ & + \frac{\mathrm{i}}{4}(\check{S}_\mu)^{pq}(\check{S}_\nu)^{rs}[\Sigma_{qp}, \Sigma_{sr}].\end{aligned} \tag{42}$$

We shall now calculate the second term on the right-handed side of (42): $\frac{\mathrm{i}}{4}(\check{S}_\mu)^{pq}(\check{S}_\nu)^{rs} \times [\Sigma_{qp}, \Sigma_{sr}] = -\frac{1}{4}[(\check{S}_\mu)_s{}^q(\check{S}_\nu)^{rs}\Sigma_{qr} - (\check{S}_\mu)^{pq}(\check{S}_\nu)^r{}_q\Sigma_{pr} - (\check{S}_\mu)_r{}^q(\check{S}_\nu)^{rs}\Sigma_{qs} + (\check{S}_\mu)^p{}_r(\check{S}_\nu)^{rs}\Sigma_{ps}]$. It can be rewritten as $\frac{1}{2}[(\check{S}_\mu)^r{}_t(\check{S}_\nu)^{ts} - (\check{S}_\nu)^r{}_t(\check{S}_\mu)^{ts}]\Sigma_{sr}$, which equals $\frac{1}{2}[\check{S}_\mu, \check{S}_\nu]^{rs}\Sigma_{sr}$. Thus, the gauge field strengths $B_{\mu\nu}$ (in the spinor representation) in (42) takes the form

$$\partial_\mu B_\nu - \partial_\nu B_\mu - \mathrm{i}[B_\mu, B_\nu] = \frac{\mathrm{i}}{2}\left(\partial_\mu\check{S}_\nu - \partial_\nu\check{S}_\mu - \mathrm{i}[\check{S}_\mu, \check{S}_\nu]\right)^{rs}\Sigma_{sr}. \tag{43}$$

This is what we have presented in (41). It is the relation of curvature tensors between the representations of tensor and spinor. This means that the gauge field tensor $B_{\mu\nu}$ that results from the commutation of covariant derivatives $\mathcal{D}_\mu^{(\mathrm{sp})}$ and $\mathcal{D}_\nu^{(\mathrm{sp})}$ (acting on the spinor fields) is actually equivalent to another gauge field tensor $\check{\Omega}_{\mu\nu}$ that results from the commutation of covariant derivatives $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ (acting on the vectors and tensors). For this reason, the gravitational Lagrangian (quadratic in curvatures) that is constructed in terms of $\check{\Omega}_{\mu\nu}$ for this Yang-Mills formulation of gravity can also be expressed in terms of $B_{\mu\nu}$ in the spinor representation.

5 Generalization of the Gravitational Gauge Theory

Now we generalize the present gravitational gauge theory in order to include the traditional Yang-Mills fields (i.e., the internal gauge interactions [31]). It is of interest that the Einstein equation of general relativity is one of the first-integral solutions to the field equation of spin-connection Yang-Mills type gauge field (since the Yang-Mills type gravitational field equation is a third-order equation of metric). But it should be noted that in the conventional Yang-Mills gauge field theory, there were no 'vielbeins' addressed. As the spin connection in the gravitational gauge theory can be expressed in terms of the vierbeins (the *vielbein* fields in four dimensions are referred to as the *vierbein* fields), one can, by analogy, conclude that the so-called 'Yang-Mills vielbeins' could also be defined, and the traditional Yang-Mills connection can be expressed in terms of such 'Yang-Mills vielbeins'. In other words, the Yang-Mills gauge interaction can be described in the formulation of vielbeins.

The question of unifying the gravitational field with other gauge fields has long been suggested [11–14] and, in the meantime, such a unification problem was always disturbing physicists. In the literature, the research results have not yet answered unambiguously whether the gravitational and the Yang-Mills fields have the same physical origin. In the present paper, we have shown that the gravity can be reconstructed as a Yang-Mills type gauge interaction when the Lorentz group is regarded as a local gauge symmetry group in the vierbein formulation. On the other hand, we suggest that the gravitational field would not be the only one which can be described in the formulation language of vielbeins. There may be other fields that can also be formulated in terms of vielbeins. Take the Yang-Mills field for example, the gauge potential, $\frac{1}{\mathrm{i}g}\lambda_{bN}\partial_\mu\zeta^{Na}$, has, as the following relation (48) indicates, satisfied spontaneously the Yang-Mills local gauge transformation. Here, the Yang-Mills gauge potential is expressed in terms of the vielbein fields (ζ^{Na} and its complex conjugate λ^{aN}), which, we shall, for brevity, call them the Yang-Mills vielbeins, to distinguish them from both the vierbeins defined on the four-dimensional spacetime introduced before and the generalized vielbeins to be introduced later in this section.

Now we consider the Yang-Mills gauge field structure by using the vielbein language and attempt to reformulate the traditional Yang-Mills gauge field theory as a vielbein field theory. Define the Yang-Mills vielbein fields ζ^{Ma} and λ_{aN} as follows

$$\zeta^{Ma}\lambda_{aN}=\delta^M{}_N,\qquad \lambda_{aN}\zeta^{Nb}=\delta_a{}^b,\qquad \zeta_M{}^a\lambda_{aN}=g_{MN},\qquad \lambda_{aN}\zeta^N{}_b=\eta_{ab},\tag{44}$$

where g_{MN} and η_{ab} can be considered to be "curved" metric and "flat" metrics, respectively, on the Yang-Mills gauge group manifold. The Latin indices $a,b,c\ldots$ and $M,N,L\ldots$ denote the matrix indices of the Yang-Mills vielbeins in the matrix representation. In an su(n) group, for example, both a and M runs over from 1 to n. If the metrics are Hermitian, then one can arrive at $\zeta^\dagger=\lambda$, i.e., $(\zeta_{Nb})^*=\lambda_{bN}$, $(\zeta^{Nb})^*=\lambda^{bN}$, $(\zeta_N{}^b)^*=\lambda^b{}_N$, and $(\zeta^N{}_b)^*=\lambda_b{}^N$. We assume that the Yang-Mills vielbein field satisfies the equation of motion $(\partial_\mu-\mathrm{i}gA_\mu)\zeta^N=0$, i.e. the component equation is given by

$$\partial_\mu\zeta^{Na}-\mathrm{i}g(A_\mu)^a{}_b\zeta^{Nb}=0,\tag{45}$$

where A_μ and g denote the Yang-Mills gauge field potential (dynamical variable) and the coupling coefficient, respectively. The indices a,b of $(A_\mu)^a{}_b$ are the matrix indices of the generators of the gauge algebras in certain representations, to which ζ^N belongs. It follows from (45) that the Yang-Mills connection is given by

$$g(A_\mu)^a{}_b=\frac{1}{\mathrm{i}}\lambda_{bN}\partial_\mu\zeta^{Na}.\tag{46}$$

Now we show that the gauge potential (46) really agrees with the local gauge transformation. Here, the gauge transformations of the Yang-Mills vielbein fields are

$$\zeta'^{Na} = U^a{}_b \zeta^{Nb}, \qquad \lambda'_{cN} = \lambda_{dN}(U^{-1})^d{}_c. \tag{47}$$

Then, the local gauge transformation of the dynamical variable of Yang-Mills field is given by

$$\begin{aligned}(A'_\mu)^a{}_c &= \frac{1}{\mathrm{i}}\lambda'_{cN}\partial_\mu \zeta'^{Na} \\ &= \frac{1}{\mathrm{i}}\lambda_{dN}(U^{-1})^d{}_c \partial_\mu \left(U^a{}_b \zeta^{Nb}\right) \\ &= U^a{}_b \frac{1}{\mathrm{i}}\lambda_{dN}\left(\partial_\mu \zeta^{Nb}\right)(U^{-1})^d{}_c + \frac{1}{\mathrm{i}}\lambda_{dN}\zeta^{Nb}(U^{-1})^d{}_c \partial_\mu U^a{}_b \\ &= U^a{}_b (A_\mu)^b{}_d (U^{-1})^d{}_c - \mathrm{i}\left(\partial_\mu U^a{}_d\right)(U^{-1})^d{}_c \\ &= \left(U A_\mu U^{-1}\right)^a{}_c - \mathrm{i}\left((\partial_\mu U)U^{-1}\right)^a{}_c. \end{aligned} \tag{48}$$

Thus, we have the well-known Yang-Mills local gauge transformation

$$A'_\mu = U A_\mu U^{-1} - \mathrm{i}(\partial_\mu U)U^{-1}. \tag{49}$$

This, therefore, means that the Yang-Mills dynamical variable, A_μ, expressed in terms of the so-called Yang-Mills vielbeins can automatically obey the local Yang-Mills gauge transformation.

We have shown that both the gravitational field theory (e.g. GR) and the Yang-Mills gauge field theory can be reconstructed in the formulation of vielbeins. This provides us with new insight into a possible way to unify these two gauge fields based on generalized vielbein fields. In order to distinguish the indices between the gravitational vielbeins and the Yang-Mills vielbeins, we choose e.g. $\underline{N}$ and $\underline{a}$ (instead of N and a used in the above) to represent the indices of the Yang-Mills vielbeins. In this paper, we impose constraints on the generalized vielbein fields $\Theta^\mu{}_r{}^{\underline{Na}}$ and $\mathcal{E}_r{}^{\mu\underline{aN}}$:

$$\Theta^\mu{}_r{}^{\underline{Ma}} \mathcal{E}^{r\nu}{}_{\underline{a}}{}^{\underline{N}} = g^{\mu\nu\underline{MN}}, \qquad \mathcal{E}^s{}_\mu{}^{\underline{bN}} \Theta^{\mu r}{}_{\underline{N}}{}^{\underline{a}} = \eta^{sr\underline{ba}}, \tag{50}$$

where $g^{\mu\nu\underline{MN}}$ and $\eta^{sr\underline{ba}}$ denote the generalized curved and flat metrics, respectively. Then we suppose that the generalized vielbeins agree with the following equations of motion

$$\begin{aligned}\left(\nabla_\mu - \mathrm{i}\mathcal{Q}_\mu\right)\Theta^{\alpha\underline{N}} &= 0, \\ \nabla_\mu \Theta^{\alpha r\underline{Na}} - \mathrm{i}(\mathcal{Q}_\mu)^r{}_s{}^{\underline{a}}{}_{\underline{b}} \Theta^{\alpha s\underline{Nb}} &= 0, \end{aligned} \tag{51}$$

where the second equation represents each component of the first equation. Here, $\mathcal{Q}_\mu$ can be referred to as the generalized spin-affine connection in contrast to the ordinary spin-affine connection addressed in the preceding sections. Multiplying (51) by generalized vielbeins $\mathcal{E}_{t\alpha\underline{cN}}$, one can arrive at

$$\begin{aligned}\mathcal{E}_{t\alpha\underline{cN}} \nabla_\mu \Theta^{\alpha r\underline{Na}} &= \mathrm{i}(\mathcal{Q}_\mu)^r{}_s{}^{\underline{a}}{}_{\underline{b}} \mathcal{E}_{t\alpha\underline{cN}} \Theta^{\alpha s\underline{Nb}} \\ &= \mathrm{i}(\mathcal{Q}_\mu)^r{}_s{}^{\underline{a}}{}_{\underline{b}} \eta_t{}^s{}_{\underline{c}}{}^{\underline{b}} \\ &= \mathrm{i}(\mathcal{Q}_\mu)^r{}_t{}^{\underline{a}}{}_{\underline{c}}. \end{aligned} \tag{52}$$

Then the dynamical variable (connection) of the generalized gauge field theory is given by

$$(\mathcal{Q}_\mu)^r{}_t{}^{\underline{a}}{}_{\underline{c}} = \frac{1}{\mathrm{i}}\mathcal{E}_{t\alpha\underline{cN}}\nabla_\mu\Theta^{\alpha r\underline{Na}}, \tag{53}$$

which is a generalization of both the Yang-Mills connection (46) and the spin-affine connection $(\check{S}_\mu)^r{}_t$.

Now we consider the generalized gauge transformation for the generalized gauge field. First we present the generalized gauge transformation of the generalized vielbein fields as follows

$$\Theta'^{\alpha r\underline{Na}} = U^r{}_s{}^{\underline{a}}{}_{\underline{b}}\Theta^{\alpha s\underline{Nb}}, \qquad \mathcal{E}'_{t\alpha\underline{cN}} = \mathcal{E}_{p\alpha\underline{dN}}(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}}. \tag{54}$$

Based on (54), the generalized gauge transformation for the dynamical variable $\mathcal{Q}'_\mu$ in the present prescription of the generalized gauge field can be obtained:

$$\begin{aligned}
(\mathcal{Q}'_\mu)^r{}_t{}^{\underline{a}}{}_{\underline{c}} &= \frac{1}{\mathrm{i}}\mathcal{E}'_{t\alpha\underline{cN}}\nabla_\mu\Theta'^{\alpha r\underline{Na}} \\
&= \frac{1}{\mathrm{i}}\mathcal{E}_{p\alpha\underline{dN}}(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}}\nabla_\mu\left(U^r{}_s{}^{\underline{a}}{}_{\underline{b}}\Theta^{\alpha s\underline{Nb}}\right) \\
&= U^r{}_s{}^{\underline{a}}{}_{\underline{b}}\frac{1}{\mathrm{i}}\mathcal{E}_{p\alpha\underline{dN}}\left(\nabla_\mu\Theta^{\alpha s\underline{Nb}}\right)(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}} + \frac{1}{\mathrm{i}}\mathcal{E}_{p\alpha\underline{dN}}\Theta^{\alpha s\underline{Nb}}(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}}\partial_\mu U^r{}_s{}^{\underline{a}}{}_{\underline{b}} \\
&= U^r{}_s{}^{\underline{a}}{}_{\underline{b}}(\mathcal{Q}_\mu)^s{}_p{}^{\underline{b}}{}_{\underline{d}}(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}} - \mathrm{i}\left(\partial_\mu U^r{}_p{}^{\underline{a}}{}_{\underline{d}}\right)(U^{-1})^p{}_t{}^{\underline{d}}{}_{\underline{c}} \\
&= \left(U\mathcal{Q}_\mu U^{-1}\right)^r{}_t{}^{\underline{a}}{}_{\underline{c}} - \mathrm{i}\left((\partial_\mu U)U^{-1}\right)^r{}_t{}^{\underline{a}}{}_{\underline{c}}.
\end{aligned} \tag{55}$$

This means that the generalized gauge transformation is given by

$$\mathcal{Q}'_\mu = U\mathcal{Q}_\mu U^{-1} - \mathrm{i}(\partial_\mu U)U^{-1}. \tag{56}$$

Thus, we have shown that the transformation (56) for the dynamical variable of the generalized gauge field is analogous to the Yang-Mills gauge transformation. It should be noted that here the matrix indices (components) of the gauge transformation (56) include both the Lorentz indices $r, s, t\ldots$ (for the gravitational vielbeins) and the Yang-Mills gauge-group generator indices $\underline{a}, \underline{b}, \underline{c}\ldots$ (for the Yang-Mills vielbeins).

As far as the dynamical variable of the generalized gauge field (53) is concerned, the gravitational field and the Yang-Mills gauge field can comply with the same gauge transformation (56), and surely obey the same kind of field equations. In order to compare this scenario with the traditional gravitational and Yang-Mills gauge field theories, we shall demonstrate that both the conventional gravitational field and the Yang-Mills gauge field can originate from the present generalized gauge field theory. If, for example, the generalized vielbeins can be factorized, i.e., they can be expressed as the products of gravitational vielbeins and Yang-Mills vielbeins:

$$\Theta^{\alpha r\underline{Na}} = \vartheta^{\alpha r}\zeta^{\underline{Na}}, \qquad \mathcal{E}_{t\alpha\underline{cN}} = e_{t\alpha}\lambda_{\underline{cN}}, \tag{57}$$

then the dynamical variable of the generalized gauge field becomes

$$\begin{aligned}
(\mathcal{Q}_\mu)^r{}_t{}^{\underline{a}}{}_{\underline{c}} &= \frac{1}{\mathrm{i}}e_{t\alpha}\lambda_{\underline{cN}}\nabla_\mu\left(\vartheta^{\alpha r}\zeta^{\underline{Na}}\right) \\
&= \frac{1}{\mathrm{i}}e_{t\alpha}\lambda_{\underline{cN}}\left(\nabla_\mu\vartheta^{\alpha r}\right)\zeta^{\underline{Na}} + \frac{1}{\mathrm{i}}e_{t\alpha}\lambda_{\underline{cN}}\vartheta^{\alpha r}\nabla_\mu\zeta^{\underline{Na}} \\
&= (\check{S}_\mu)^r{}_t\eta^{\underline{a}}{}_{\underline{c}} + g(A_\mu)^{\underline{a}}{}_{\underline{c}}\eta^r{}_t,
\end{aligned} \tag{58}$$

where $\eta^{\underline{a}}{}_{\underline{c}} = \delta^{\underline{a}}{}_{\underline{c}}$ and $\eta^r{}_t = \delta^r{}_t$. From the relation (58), we can see that the generalized gauge field $(\mathcal{Q}_\mu)^r{}_t{}^{\underline{a}}{}_{\underline{c}}$ can be decomposed into two parts: $(\check{S}_\mu)^r{}_t\eta^{\underline{a}}{}_{\underline{c}}$ and $g(A_\mu)^{\underline{a}}{}_{\underline{c}}\eta^r{}_t$, which we can interpret as the gravitational gauge field and the Yang-Mills gauge field, respectively. It would be easy to construct the Lagrangian densities and then to obtain the field equations for the generalized vielbein fields, where both the gravitational gauge field equation and the Yang-Mills equation are involved within the same framework.

6 Discussions and Concluding Remarks

As is well known, in the formulation of metric and Levi-Civita connection, the dynamical variable of gravitational field is the metric. In the formulation of vierbein and spin-affine connection, however, the dynamical variable of the gravitational field is the spin-affine connection, and then GR becomes a Yang-Mills type gauge field theory. We have considered the gauge approach to gravitation and suggested a gravitational field equation within the framework of spin-connection gauge theory for torsion-free gravity. One of our aims in this paper is to develop a theory of local Lorentz-group spin-connection gauge field as elementary a point of view as possible and to derive it as a natural consequence of the Yang-Mills gauge field theory. The Yang-Mills type gravity theory has been extended to include the other Yang-Mills type interactions. By following the procedure of the gravitational gauge theory, the Yang-Mills vielbeins were defined and then used to construct the Yang-Mills connections. This enables us to suggest a possible route to unify gravity and Yang-Mills fields in the framework of the generalized vielbeins (this might be referred to as the two-fold vielbeins of composite spacetime manifold, the affine connection of which has both the gravitational indices $r, s, t \ldots$ and the Yang-Mills gauge group indices $\underline{a}, \underline{b}, \underline{c}, \ldots$).

We have shown that the cosmological constant, λ, in the Yang-Mills type gravity theory, where the spin-affine connection becomes the dynamical variable of the local Lorentz-group gauge field, has already been eliminated (simply because the dynamical equation is a third-order differential equation of metric), and actually makes no gravitational contributions. However, as the Einstein field equation is a first-integral solution to the Yang-Mills gravitational gauge field equation, an integration constant term (say, $\Lambda g_{\mu\nu}$) that can appear in the Einstein field equation would naturally play an equivalent role of the cosmological constant, but its value depends on the realistic physical conditions (such as the initial or boundary condition) of the gravitating system itself. In other words, the present gauge field equation of Yang-Mills type can automatically exhibit an effective cosmological constant Λ, though the infinite (or very large) quantum vacuum energy density (expressed by λ) no longer makes any contributions to gravitation. Obviously, the physical meaning of the present equivalent cosmological constant Λ is no longer the density of vacuum energy or dark energy, either. Additionally, the idea that we view the practical (observed) cosmological constant term as an integration constant term could naturally interpret the observed cosmological constant value that is close to the critical density: specifically, the integration constant of the solutions (to the Yang-Mills type field equation or, equivalently, to the SKY equation) depends on the cosmological characteristic scale, if we apply the solutions to the dynamical equations of the cosmic evolution and structure. Thus, the equivalent cosmological constant value Λ would be related closely to the cosmic large-scale structure or the boundary condition (e.g. $\Lambda \simeq 1/a^2$ with a being the typical scale of the Universe). This makes the observed cosmological constant Λ close to the critical density, as demonstrated in observational cosmology [1–3].

Although in the literature many researchers have investigated the gravity with torsion [8, 9], yet less attention was paid to the Riemann-Cartan geometry [32–34] with the spin connection being the dynamical variable. One of such cases is the Yang-Mills type equation for the complex-metric gravity, where both torsion and curvature emerge (since the contorsion that is a tensor in the theory of real and symmetric metric is no longer a tensor in the complex-metric theory of gravitation, the contorsion cannot vanish, and as a result, the torsion is necessary in the complex-metric theory). We believe that it is worth extending the present torsion-free gravitational gauge field theory to the torsional gravity. We hope that it could open a possible research field for considering the gravitational contributions of spin (including the graviton spin contribution to gravitation [35]) by means of other ways than the Cartan gravity theory [32–34].

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